TAUTOLOGICAL ALGEBRA OF THE MODULI STACK OF SEMISTABLE BUNDLES OF RANK 2 ON A GENERAL CURVE

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Abstract. Our aim in this paper is to determine the tautological algebra generated by the cohomology classes of the Brill-Noether loci in the rational cohomology of the moduli stack $\mathcal{U}_C(n,d)$ of semistable bundles of rank n and degree d. When C is a general smooth projective curve of genus $g \geq 2$, $d = 2g-2$, the tautological algebra of $\mathcal{U}_C(2, 2g-2)$ (resp. the moduli stack $\mathcal{SU}_C(2,\mathcal{L})$ of semistable bundles of rank 2 and determinant $\mathcal{L}$ with $deg(\mathcal{L}) = 2g-2$) is generated by the divisor classes (resp. the class of the Theta divisor Θ).

Contents

1. Introduction

Suppose C is a complex smooth projective curve of genus g. The Jacobian variety $J_d(C)$ parametrises isomorphism classes of degree d line bundles on C. The classical Brill-Noether subvarieties W^r_d of $J_d(C)$ parametrise line bundles with at least $r+1$ linearly independent sections. The questions on non-emptiness, dimension, and irreducibility of the loci have been classically studied (cf. [ACGH, Chapter IV], [Gf-Hr 1] and [Su]). In other direction, the classical Poincaré formula expresses the cohomological classes of W^0_i, in terms of the Theta divisor on $J(C)$:

$$[W^0_i] = \frac{1}{(g-i)!}[\Theta]^{g-i} \in H^*(J(C), \mathbb{Q}).$$

Here we identify $J_d(C) \cong J_0(C) = J(C)$ (cf. [ACGH, Chapter 1, §5; p. 25]).

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A similar formula holds amongst the cohomology classes of W^r_d, for varying r in $J_d(C)$, for a general smooth curve C (see Theorem 6.5) :

$$[W^r_d] = \prod_{\alpha=0}^{r} \frac{\alpha!}{(g-d+r+\alpha)!}.[\Theta_d]^{g-\rho}.$$

Here $\rho = \dim(W^r_d)$. When $\rho = 0$, the above formula is known as the Castelnuovo formula or the Porteous formula, in general.

For $n, d \geq 1$ we denote by $\mathcal{U}_C(n,d)$ the moduli stack of semistable vector bundles of rank n and degree d. For $\mathcal{L} \in \operatorname{Pic}(C)$ we denote by $\mathcal{SU}_C(n,\mathcal{L})$ the moduli stack of semistable vector bundles of rank n and determinant $\mathcal{L}$. Their coarse moduli spaces $\mathcal{U}_C(n,d)$ and $\mathcal{SU}_C(n,\mathcal{L})$ have been widely studied. The Brill-Noether loci have been similarly defined and investigated, and significant results have been obtained (cf. [Bg 1], [Bg 2], [Br-Gz-Ne], [Me 1], [Me 2] and [Su]). More recent developments on non-emptiness of the Brill-Noether loci can be found in [La-Ne-St], [La-Ne-Pr], [La-Ne 1], [La-Ne 2] and [La-Ne 3].

Some questions on the cohomology classes have been raised by C. S. Seshadri and N. Sundaram in [Su, p. 176]. As pointed out by P. Newstead, the virtual cohomology classes can be worked out in terms of the known generators of the cohomology of moduli space when n and d are coprime, using the determinantal structure of the Brill-Noether loci. But it is difficult in general to determine whether a polynomial in the generators gives a non-zero cohomology class. Even in rank two, where the generators for the relations are well known, this is difficult. In general the questions are open and in this paper, we illustrate a method to address these questions. We consider rank two situation, and when the degree $d = 2g-2$.

Our aim in this paper is to obtain a Poincaré type relation amongst the cohomological classes of the geometric Brill-Noether loci, when the curve C is general. As a first step, in [Mk], when $g = 1$ the relations were found to be similar to the Poincaré relations. The moduli spaces $U_C(2, 2g-2)$ and $SU_C(2,\mathcal{L})$ are singular varieties, and the cohomology class of a Brill-Noether locus is well-defined in the cohomology of the moduli stack $H^*(\mathcal{U}_C(2, 2g-2), \mathbb{Q})$ (resp. $H^*(\mathcal{SU}_C(2,\mathcal{L}), \mathbb{Q})$) (see §8)).

We show the following:

Theorem 1.1. *Suppose C is a general smooth projective curve of genus g, and $g \geq 2$. The cohomology class of a Brill-Noether locus on the moduli stack $\mathcal{U}_C(2, 2(g-1))$ can be expressed as a polynomial on the divisor classes, with rational coefficients.*

See Theorem 9.6.

Similarly, consider the moduli stack $\mathcal{SU}_C(2,\mathcal{L})$ of semistable bundles of rank r and fixed determinant $\mathcal{L}$ of degree $2g-2$ on C. Then we obtain the following:

Corollary 1.2. *The cohomology class of a Brill-Noether locus $\widetilde{W^{r,\mathcal{L}}_{2,2(g-1)}}$ in the moduli stack $\mathcal{SU}_C(2,\mathcal{L})$ is a polynomial expression on the class of the Theta divisor, with rational coefficients. In other words, the tautological algebra generated by the Brill-Noether loci is generated by the class of the theta divisor.*

See Corollary 9.7.

The nontriviality of the above algebra follows from [La-Ne-Pr].

The key idea is to relate the Brill-Noether loci on the moduli space with the Brill-Noether loci on the Jacobian variety of a general spectral curve. We utilise the rational map obtained in [Be-Na-Ra] from the Jacobian of a general spectral curve to the moduli space $U_C(2, 2(g-1))$. Another ingredient is to note that the Hodge conjecture holds for the Jacobian of a general spectral curve, via a computation of the Mumford-Tate group [Bi]. We also use the fact that the moduli stacks are quotients stacks [Go]. Hence their cohomology is the equivariant cohomology of (an open subset of) Quot scheme. This enables us to define the Brill-Noether classes in the cohomology of the moduli stack. See §8.

For moduli stacks $\mathcal{U}_C(n, d)$, when $n > 2$ and $d \neq 2g - 2$, the same proofs and techniques hold. However to obtain the final conclusion, we need to have that the Hodge conjecture holds for Jacobian of a higher degree general spectral curve (cf. [Ar], for unramified coverings). The proofs employed in Theorem 1.1 will then also hold for higher rank moduli spaces. For degree $d \neq 2g-2$, we need to know that the divisor classes on the Jacobian of spectral curve descend on the moduli space (see Lemma 9.5). P. Newstead informed us that the descent of divisor classes is known in several other cases too.

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2. Notations

All the varieties are defined over complex numbers.

(1) $\mathcal{U}_C(n, d)$ denotes the moduli stack of S- equivalence classes of semistable bundles of rank r and degree d over C. $U_C(n, d)$ denotes its coarse moduli scheme.
(2) $\mathcal{SU}_C(n, \mathcal{L})$ denotes the moduli stack of S-equivalence classes of semistable bundles of rank r and fixed determinant $\mathcal{L}$ of degree d over C. $SU_C(n, \mathcal{L})$ denotes its coarse moduli scheme.
(3) $J(C)$ is the Jacobian variety of isomorphism classes of line bundles of degree 0 over C.
(4) $J_d(C)$ is the isomorphism classes of line bundles of degree d over C.
(5) $\mathcal{O}(D)$ denotes the line bundle corresponding to a divisor D on C.
(6) Given a closed subvariety $W \subset X$, $[W]$ denotes the cohomology class in the integral or rational cohomology group of X.

3. Spectral curves and Moduli Spaces

In this section we recall the construction of spectral curve from [Be-Na-Ra] which will be needed in this paper.

3.1. **Spectral curve.** Let C be a smooth projective curve of genus $g \geq 2$ defined over complex numbers. Fix $n \geq 1$. Let L be a line bundle on C and $s = (s_k)$ be sections of L^k for $k = 1, 2, \cdots, n$. Let $\pi : \mathbb{P}(\mathcal{O}_C \oplus L^*) \to C$ be the natural projection map and $\mathcal{O}(1)$ be the relatively ample bundle. Then $\pi_*(\mathcal{O}(1))$ is naturally isomorphic to $\mathcal{O}_C \oplus L^*$ and therefore has a canonical section. This provides a section of $\mathcal{O}(1)$ denoted by y. By projection formula we have:

$$\pi_*(\pi^* L \otimes \mathcal{O}(1)) \cong L \otimes \pi_*(\mathcal{O}(1)) \cong L \otimes (\mathcal{O}_C \oplus L^*) = L \oplus \mathcal{O}_C.$$

Therefore $\pi_*(\pi^* L \otimes \mathcal{O}(1))$ also has a canonical section and we denote the corresponding section of $\pi^* L \otimes \mathcal{O}(1)$ by x. Consider the section

$$x^n + (\pi^* s_1) y x^{n-1} + \cdots + (\pi^* s_n) y^n \tag{1}$$

of $\pi^* L^n \otimes \mathcal{O}(n)$. Zero scheme of this section is a subscheme of $\mathbb{P}(\mathcal{O}_C \oplus L^*)$ and is called *spectral curve* of the given curve C and is denoted by $\widetilde{C}_s$ or $\widetilde{C}$ in short. Let $\pi : \widetilde{C} \to C$ be the restriction of the natural projection $\pi : \mathbb{P}(\mathcal{O}_C \oplus L^*) \to C$. It can be checked that $\pi : \widetilde{C} \to C$ is finite and its fiber over any point $c \in C$ is a subscheme of $\mathbb{P}^1$ given by

$$x^n + a_1 y x^{n-1} + \cdots + a_n y^n = 0,$$

where (x, y) is a homogeneous co-ordinate system and a_i is the value of s_i at c.
Let $\widetilde{g}$ be the genus of $\widetilde{C}$. As $\pi_*(\mathcal{O}_{\widetilde{C}}) \cong \mathcal{O}_C \oplus L^{-1} \oplus \cdots \oplus L^{-(n-1)}$, we have the following relation between genus $\widetilde{g}$ of the spectral curve $\widetilde{C}$ and genus g of C using Riemann-Roch theorem

$$1 - \widetilde{g} = \chi(\widetilde{C}, \mathcal{O}_{\widetilde{C}}) = \chi(C, \pi_*(\mathcal{O}_{\widetilde{C}})) = \sum_{i=0}^{n-1} \chi(C, L^{-i}) = -(\deg L) \cdot \frac{n(n-1)}{2} + n(1-g).$$

Hence we have:

$$\widetilde{g} = (\deg L) \cdot \frac{n(n-1)}{2} + n(g-1) + 1. \tag{2}$$

Moreover if we take the line bundle L to be of degree $2g - 2$, say the canonical line bundle K_C for example, then from (2) the genus $\widetilde{g}$ of the corresponding spectral curve $\widetilde{C}$ is given by:

$$\widetilde{g} = n^2(g-1) + 1 = \dim U_C(n, d). \tag{3}$$

3.2. **Spectral curve and moduli space of semistable bundles.** Here we relate the spectral curve $\widetilde{C}$ with the moduli space of semistable bundles of fixed rank and degree over C. Consider the following theorem.

Theorem 3.1. *Let C be any curve and L any line bundle on C. Let $(s) = ((s_i)) \in \Gamma(L) \oplus \Gamma(L^2) \oplus \cdots \oplus \Gamma(L^n)$ be so chosen such that the corresponding spectral curve $\widetilde{C}_s$ is integral, smooth and non-empty. Then there is a bijective correspondence between isomorphism classes of line bundles on $\widetilde{C}_s$ and isomorphism classes of pairs (E, ϕ) where E is a vector bundle of rank n and $\phi : E \to L \otimes E$ a homomorphism with characteristic coefficients s_i.*

Proof. See [Be-Na-Ra, Proposition 3.6, Remark 3.1,3.5 and 3.8; p. 172-174]. ∎